AN ALGEBRAIC PROOF OF CYCLIC SUM FORMULA
FOR MULTIPLE ZETA VALUES

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ABSTRACT. We introduce an algebraic formulation of cyclic sum formulas for multiple zeta values and for multiple zeta-star values. We also present an algebraic proof of cyclic sum formulas for multiple zeta values and for multiple zeta-star values by reducing them to Kawashima relation.

1. INTRODUCTION

The only proof of cyclic sum formula (abbreviated as CSF) for multiple zeta values (resp. for multiple zeta-star values) which has been known so far is by using partial fraction expansions, which appears in Hoffman-Ohno [4] (resp. Ohno-Wakabayashi [9]). We have interested in the other proofs of CSF, especially by discussing relationships between CSF and the other known relations for multiple zeta values such as regularized double shuffle relation ([6]), associator relation ([2]), etc. In such a motivation, we introduce herein an algebraic formulation of CSF for multiple zeta values and CSF for multiple zeta-star values, and give an algebraic proof of CSF by reducing it to (the linear part of) Kawashima relation ([7]).

For $k_1 > 1, k_2, \ldots, k_n \geq 1$, the multiple zeta value (abbreviated as MZV) is a real number defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

and the multiple zeta-star value (abbreviated as MZSV) is defined by the convergent series

$$\zeta^{\star}(k_1, k_2, \ldots, k_n) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

We call the number $k_1 + \cdots + k_n$ weight and n depth. If $n = 1$, MZV and MZSV coincide and are known as a Riemann zeta value.

Throughout the present paper, we employ the algebraic setup introduced by Hoffman [3]. Let $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y, and let $\mathfrak{H}^1$ and $\mathfrak{H}^0$ denote the subalgebras $\mathbb{Q} + \mathfrak{H}y$ and $\mathbb{Q} + x\mathfrak{H}y$, respectively. We define the $\mathbb{Q}$-linear map $Z \colon \mathfrak{H}^0 \to \mathbb{R}$ by $Z(1) = 1$ and

$$Z(x^{k_1-1}yx^{k_2-1}y \cdots x^{k_n-1}y) = \zeta(k_1, k_2, \ldots, k_n).$$

We also define the $\mathbb{Q}$-linear map $\overline{Z} \colon \mathfrak{H}^0 \to \mathbb{R}$ by $\overline{Z}(1) = 1$ and

$$\overline{Z}(x^{k_1-1}yx^{k_2-1}y \cdots x^{k_n-1}y) = \zeta^{\star}(k_1, k_2, \ldots, k_n).$$

The degree (resp. degree with respect to y) of a word is the weight (resp. the depth) of the corresponding MZV or MZSV. Let $d \colon \mathfrak{H}^1 \to \mathfrak{H}^1$ be a $\mathbb{Q}$-linear map defined by

$$d(wy) = \gamma(w)y$$

for $w \in \mathfrak{H}$, where γ is the automorphism on $\mathfrak{H}$ given by

$$\gamma(x) = x, \ \gamma(y) = x + y.$$

It is well-known that the identity $\overline{Z} = Zd$ holds.

For $k_1, \ldots, k_l \geq 1$ with some $k_q > 1$, CSF for MZV's

$$(1) \quad \sum_{j=1}^{l} \sum_{i=1}^{k_j - 1} \zeta(k_j - i + 1, k_{j+1}, \ldots, k_l, k_1, \ldots, k_{j-1}, i) = \sum_{j=1}^{l} \zeta(k_j + 1, k_{j+1}, \ldots, k_l, k_1, \ldots, k_{j-1})$$

is proven in Hoffman-Ohno [4] by means of partial fraction expansions and CSF for MZSV's

$$(2) \qquad \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \zeta^{\star}(k_j - i + 1, k_{j+1}, \ldots, k_l, k_1, \ldots, k_{j-1}, i) = k\zeta(k+1),$$

where $k = k_1 + \cdots + k_l$, in Ohno-Wakabayashi [9] in a similar way. Hoffman and Ohno also introduced in [4] an algebraic expression of CSF for MZV's. They formulated CSF using two cyclic derivatives C_w and $\overline{C}_w$ on $\mathfrak{H}$, which states as follows.

A cyclic derivative $\phi_\bullet \colon \mathfrak{H} \to \mathrm{End}(\mathfrak{H})$ is defined by a $\mathbb{Q}$-linear map with the property

$$\phi_{w_1 w_2}(W) = \phi_{w_1}(w_2 W) + \phi_{w_2}(W w_1)$$

for any $w_1, w_2, W \in \mathfrak{H}$. Such a cyclic derivative is uniquely determined by ϕ_x and ϕ_y. Let $C_\bullet$ (resp. $\overline{C}_\bullet$) be a cyclic derivative defined by $C_x = 0, C_y = L_x R_y$ (resp. $\overline{C}_x = L_x R_y, \overline{C}_y = 0$), where, for $w \in \mathfrak{H}$, the map L_w (resp. R_w) is a $\mathbb{Q}$-linear map, called left- (resp. right-) multiplication, defined by $L_w(W) = wW$ (resp. $R_w(W) = Ww$) for any $W \in \mathfrak{H}$. Let $\check{\mathfrak{H}}^1$ be a subvector space of $\mathfrak{H}^1$ generated by words of $\mathfrak{H}^1$ except for powers of y. Then,

Theorem 1.1 (Hoffman-Ohno). *For any word $w \in \check{\mathfrak{H}}^1$, we have*

$$Z((\overline{C}_w - C_w)(1)) = 0.$$

2. Main results

In this section, we give an algebraic formulation of CSF for MZV's, which the authors have been inspired by works for double Poisson algebras introduced in [1]. The method of the formulation is a little different from Hoffman-Ohno's. Then we present an algebraic proof of CSF for MZV's by reducing it to (the linear part of) Kawashima relation.

Let $n \geq 1$. We denote an action of $\mathfrak{H}$ on $\mathfrak{H}^{\otimes(n+1)}$ by "$\diamond$", which is defined by

$$a \diamond (w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \otimes \cdots \otimes w_n \otimes a w_{n+1},$$

$$(w_1 \otimes \cdots \otimes w_{n+1}) \diamond b = w_1 b \otimes w_2 \otimes \cdots \otimes w_{n+1}.$$

The action $\diamond$ is a $\mathfrak{H}$-bimodule structure on $\mathfrak{H}^{\otimes(n+1)}$.

Let $z = x + y$. We define the $\mathbb{Q}$-linear map $\mathcal{C}_n \colon \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\mathcal{C}_n(x) = x \otimes z^{\otimes(n-1)} \otimes y, \ \mathcal{C}_n(y) = -(x \otimes z^{\otimes(n-1)} \otimes y)$$

and

$$(3) \qquad \mathcal{C}_n(ww') = \mathcal{C}_n(w) \diamond w' + w \diamond \mathcal{C}_n(w')$$

for any $w, w' \in \mathfrak{H}$. The map $\mathcal{C}_n$ is well-defined because of the identities

$$a \diamond (b \diamond w) = ab \diamond w, \ (w \diamond a) \diamond b = w \diamond ab,$$

where $a, b \in \mathfrak{H}, w \in \mathfrak{H}^{\otimes(n+1)}$. We also find that $\mathcal{C}_n(1) = 0$ by putting $w = w' = 1$ in (3).

Let $M_n \colon \mathfrak{H}^{\otimes(n+1)} \to \mathfrak{H}$ denote the multiplication map, i.e.,

$$M_n(w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \cdots w_{n+1},$$

and let $\rho_n = M_n \mathcal{C}_n$ $(n \geq 1)$. Then our main theorem states

Theorem 2.1. *For $n \geq 1$, we have $\rho_n(\check{\mathfrak{H}}^1) \subset \ker Z$.*

The theorem contains Theorem 1.1 because of the following proposition.

Proposition 2.2. *For any $w \in \mathfrak{H}$, we have $\rho_1(w) = (\overline{C}_w - C_w)(1)$.*

To prove the proposition, we firstly show the following lemma.

Lemma 2.3. *For cyclically equivalent words $w, w' \in \mathfrak{H}$, we have $\rho_1(w) = \rho_1(w')$.*

Proof. Let $u_1, \ldots, u_l \in \{x, y\}$ and $\mathrm{sgn}(u) = 1$ or -1 according to $u = x$ or y. Since

$$\mathcal{C}_1(u) = \mathrm{sgn}(u)(x \otimes y)$$

for $u \in \{x, y\}$, we have

$$\mathcal{C}_1(u_1 \cdots u_l) = \sum_{j=1}^{l} u_1 \cdots u_{j-1} \diamond \mathcal{C}_1(u_j) \diamond u_{j+1} \cdots u_l$$

$$= \sum_{j=1}^{l} \mathrm{sgn}(u_j)(xu_{j+1} \cdots u_l \otimes u_1 \cdots u_{j-1}y),$$

where we assume $u_1 \cdots u_{j-1} = 1$ if $j = 1$ and $u_{j+1} \cdots u_l = 1$ if $j = l$. Therefore we have

$$\rho_1(u_1 \cdots u_l) = \sum_{j=1}^{l} \mathrm{sgn}(u_j)xu_{j+1} \cdots u_l u_1 \cdots u_{j-1}y.$$

Since the right-hand side is cyclically equivalent for a word $u_1 \cdots u_l$, we conclude the lemma. □

Proof of Proposition 2.2. Let $z_k = x^{k-1}y$ $(k \geq 1)$. It suffices to show the identity for words $w = z_{k_1} \cdots z_{k_l}$ and x^q $(q \geq 1)$ because of Lemma 2.3.

If $w = x^q$, we easily calculate

$$C_w(1) = 0, \ \overline{C}_w(1) = \rho_1(w) = qz_{q+1},$$

and hence the proposition holds.

When $w = z_{k_1} \cdots z_{k_l}$, Hoffman and Ohno showed in [4] that

$$(4) \quad (\overline{C}_w - C_w)(1) = \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_j}.$$

To prove the proposition, we show that $\rho_1(w)$ equals the right-hand side of this identity.

By definition of $\mathcal{C}_1$, we calculate

$$\mathcal{C}_1(z_k) = \sum_{j=1}^{k-1} z_{k-j+1} \otimes z_j - x \otimes z_k.$$

Therefore we have

$$\mathcal{C}_1(w) = \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} \otimes z_{k_1} \cdots z_{k_{j-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} \otimes z_{k_1} \cdots z_{k_j},$$

and hence, $\rho_1(w)$ equals the right-hand side of (4). □

We find that the operator ρ_1 induces CSF for MZV's because of Proposition 2.2 and Theorem 1.1. According to Lemma 2.3 and Proposition 2.2, we also find that Theorem 1.1 holds if $w \in \check{\mathfrak{H}}$, where $\check{\mathfrak{H}}$ is a subvector space of $\mathfrak{H}$ generated by words of $\mathfrak{H}$ except for powers of x and powers of y.

To prove our main theorem (Theorem 2.1), we use the linear part of Kawashima relation introduces in [7], which is stated as follows. Let φ be the automorphism on $\mathfrak{H}$ given by

$$\varphi(x) = x + y, \ \varphi(y) = -y.$$

The product "$*$" stands for the harmonic product on $\mathfrak{H}^1$ introduced in Hoffman [3], which is known to be associative and commutative. Under these notations, the linear part of Kawashima relation for MZV's states

Theorem 2.4 (Kawashima). $L_x\varphi(\mathfrak{H}y * \mathfrak{H}y) \subset \ker Z.$

The theorem is used in [12] to prove the quasi-derivation relation for MZV's. To prove Theorem 2.1 by reducing CSF for MZV's to Theorem 2.4, we should show

Proposition 2.5. *For $n \geq 1$, we have $\rho_n(\check{\mathfrak{H}}^1) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$.*

This proposition holds because of the Key Proposition and the lemma below. Let $A_0 = 1$ and $A_j = z^{j-1}y$ for $j \geq 1$, where $z = x + y$.

Key Proposition 2.6. *For $k_1 \geq n, k_2, \ldots, k_l \geq 1$, we have*

$$\varphi L_x^{-1} \rho_n(A_{k_1+\cdots+k_l-n+1} - A_{k_1-n+1}A_{k_2} \cdots A_{k_l}) = \sum_{m=2}^{l} \frac{(-1)^{l-m}}{m} \sum_{j=1}^{l} \sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\ldots,\alpha_m \geq 1}} H(j, \alpha_1, \ldots, \alpha_m).$$

Here, $H(j, \alpha_1, \ldots, \alpha_m)$ is given by

$$H(j, \alpha_1, \ldots, \alpha_m) = z_{k_j} \cdots z_{k_{\alpha_1+j-1}} * z_{k_{\alpha_1+j}} \cdots z_{k_{\alpha_1+\alpha_2+j-1}} * \cdots * z_{k_{\alpha_1+\cdots+\alpha_{m-1}+j}} \cdots z_{k_{\alpha_1+\cdots+\alpha_m+j-1}},$$

where the subscripts of k's of the right-hand side are viewed as numbers modulo l $(\in \{1, \ldots, l\})$.

Proof. First we give some notations. Let $U = U(k_1, \ldots, k_l)$ be the set of tuples consisting of at most l components, each of which is the sum of some of $k_1, \ldots, k_l$, such that each k_i occurs in exactly one of the components. For example,

$$U(k_1, k_2) = \{(k_1, k_2), (k_2, k_1), (k_1 + k_2)\},$$

$$U(k_1, k_2, k_3) = \{(k_1, k_2, k_3), (k_1, k_3, k_2), (k_2, k_1, k_3), (k_2, k_3, k_1), (k_3, k_1, k_2), (k_3, k_2, k_1),$$

$$(k_1 + k_2, k_3), (k_2 + k_3, k_1), (k_3 + k_1, k_2), (k_1, k_2 + k_3), (k_2, k_3 + k_1), (k_3, k_1 + k_2), (k_1 + k_2 + k_3)\},$$

and so on. For $1 \leq i \leq l$, let $I_i = I_i(k_1, \ldots, k_l)$ be the set of tuples in U such that k_i occurs in a component that lies to the left of the one containing K_{i+1}. For example,

$$I_1(k_1, k_2) = \{(k_1, k_2)\},$$
$$I_2(k_1, k_2) = \{(k_2, k_1)\},$$
$$I_1(k_1, k_2, k_3) = \{(k_1, k_2, k_3), (k_1, k_3, k_2), (k_3, k_1, k_2), (k_1, k_2 + k_3), (k_1 + k_3, k_2)\},$$
$$I_2(k_1, k_2, k_3) = \{(k_2, k_3, k_1), (k_2, k_1, k_3), (k_1, k_2, k_3), (k_2, k_3 + k_1), (k_2 + k_1, k_3)\},$$
$$I_3(k_1, k_2, k_3) = \{(k_3, k_1, k_2), (k_3, k_2, k_1), (k_2, k_3, k_1), (k_3, k_1 + k_2), (k_3 + k_2, k_1)\},$$

and so on. We also define $W : U \to \mathfrak{H}$ by

$$W(k_1, \ldots, k_l) = z_{k_1} \cdots z_{k_l}$$

for $k_1, \ldots, k_l \geq 1$. Then we find

$$(5) \qquad \qquad \sum_{\mathbf{k} \in U} W(\mathbf{k}) = z_{k_1} * \cdots * z_{k_l}$$

and

$$(6) \qquad \qquad U \backslash \{(k_1 + \cdots + k_l)\} = \bigcup_{i=1}^{l} I_i.$$

Since

$$\varphi L_x^{-1} \rho_n(A_{k_1+\cdots+k_l-n+1} - A_{k_1-n+1} A_{k_2} \cdots A_{k_l}) = z_{k_1+\cdots+k_l} + (-1)^l \sum_{j=1}^{l} z_{k_j} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}},$$

it suffices to show

$$(7) \quad z_{k_1+\cdots+k_l} + (-1)^l \sum_{j=1}^{l} z_{k_j} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} = \sum_{m=2}^{l} \frac{(-1)^{l-m}}{m} \sum_{j=1}^{l} \sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\ldots,\alpha_m \geq 1}} H(j, \alpha_1, \ldots, \alpha_m),$$

where the subscripts of k's of the right-hand side are viewed as numbers modulo l $(\in \{1, \ldots, l\})$.

Put $N^{(l)} = \{n \in \mathbb{N} \mid 1 \leq n \leq l\}$, $N_j^{(l)} = N^{(l)} \backslash \{j\}$ $(1 \leq j \leq l)$ and $A = \{\alpha_1 + \cdots + \alpha_s + j \mid 1 \leq s < m\}$ for a fixed $(\alpha_1, \ldots, \alpha_m)$ with $\alpha_1 + \cdots + \alpha_m = l, \alpha_1, \ldots, \alpha_m \geq 1$. Expanding the harmonic products, we have

$$H(j, \alpha_1, \ldots, \alpha_m) = \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash A} I_r} W(\mathbf{k}).$$

Hence, we obtain

$$\sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\ldots,\alpha_m \geq 1}} H(j, \alpha_1, \ldots, \alpha_m) = \sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\ldots,\alpha_m \geq 1}} \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash A} I_r} W(\mathbf{k})$$

$$= \sum_{\substack{S \subset N_j^{(l)} \\ |S|=m-1}} \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash S} I_r} W(\mathbf{k}).$$

Adding up from $j = 1$ to l, we obtain

$$\sum_{j=1}^{l} \sum_{\substack{S \subset N_j^{(l)} \\ |S|=m-1}} \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \setminus S} I_r} W(\mathbf{k}) = m \sum_{\substack{T \subset N^{(l)} \\ |T|=m}} \sum_{\mathbf{k} \in \bigcap_{r \in N^{(l)} \setminus T} I_r} W(\mathbf{k})$$

$$= m \sum_{0 < i_1 < \cdots < i_{l-m} \leq l} \sum_{\mathbf{k} \in \bigcap_{p=1}^{l-m} I_{i_p}} W(\mathbf{k})$$

The reason why the middle term of the last equation is multiplied by m is because $m - 1$ elements belonging in S have been already removed from $N_j^{(l)}$ at the third summation of the first term and there are m ways to get rid $N^{(l)}$ of the additional element in T. Owing to the above equation, we obtain

$$\sum_{m=1}^{l-1} \frac{(-1)^{l-m}}{m} \sum_{j=1}^{l} \sum_{\substack{\alpha_1 + \cdots + \alpha_m = l \\ \alpha_1, \ldots, \alpha_m \geq 1}} H(j, \alpha_1, \ldots, \alpha_m)$$

$$= \sum_{m=1}^{l-1} (-1)^{l-m} \sum_{0 < i_1 < \cdots < i_{l-m} \leq l} \sum_{\mathbf{k} \in \bigcap_{p=1}^{l-m} I_{i_p}} W(\mathbf{k})$$

$$= \sum_{m=1}^{l} (-1)^{m} \sum_{0 < i_1 < \cdots < i_m \leq l} \sum_{\mathbf{k} \in \bigcap_{p=1}^{m} I_{i_p}} W(\mathbf{k})$$

$$= - \sum_{\mathbf{k} \in \bigcup_{i=1}^{l} I_i} W(\mathbf{k}).$$

The last equality is by the inclusion-exclusion property. Because of (6) and (5), we obtain

$$- \sum_{\mathbf{k} \in \bigcup_{i=1}^{l} I_i} W(\mathbf{k}) = - \sum_{\mathbf{k} \in U \setminus \{(k_1 + \cdots + k_l)\}} W(\mathbf{k})$$

$$= z_{k_1 + \cdots + k_l} - \sum_{\mathbf{k} \in U} W(\mathbf{k})$$

$$= z_{k_1 + \cdots + k_l} - z_{k_1} * \cdots * z_{k_l}.$$

Therefore we conclude (7). □

We also need the following lemma.

Lemma 2.7. *The set* $\{A_{k_1 + \cdots + k_l} - A_{k_1} \cdots A_{k_l} \mid k_1, \ldots, k_l \geq 1, l \geq 1\}$ *is a set of bases of* $\check{\mathfrak{H}}^1$.

Proof. Since the indeterminates y and $z(= x + y)$ can be generators of $\mathfrak{H}$, the set $X = \{A_{k_1} \cdots A_{k_l} \mid k_1, \ldots, k_l \geq 1\}$ is a set of bases of $\mathfrak{H}^1$. For each $k \geq 1$, the dimension of the space of weight k generated by the set $Y = \{A_{k_1} \cdots A_{k_l} - A_{k_1 + \cdots + k_l} \mid k_1, \ldots, k_l \geq 1\}$ is one less than the space of weight k generated by X. Also we find that any power of y cannot be expressed by elements of Y. Therefore we conclude the lemma. □

Thus we obtain Proposition 2.5, and hence Theorem 2.1 because of Theorem 2.4.

3. FOR MZSV'S

In §2, we exploited a new algebraic formulation to prove CSF for MZV's by reducing it to Kawashima relation. In this section, we describe an algebraic formulation and a proof of CSF for MZSV's.

As in the previous sections, let z be $x + y$ and γ the automorphism on $\mathfrak{H}$ given by

$$\gamma(x) = x, \; \gamma(y) = x + y.$$

We notice that γ^{-1} is also the automorphism on $\mathfrak{H}$ given by

$$\gamma^{-1}(x) = x, \; \gamma^{-1}(y) = y - x.$$

We define the $\mathbb{Q}$-linear map $\overline{\mathcal{C}}_n \colon \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\overline{\mathcal{C}}_n(x) = x \otimes y^{\otimes n}, \; \overline{\mathcal{C}}_n(y) = -(x \otimes y^{\otimes n})$$

and
$$\overline{\mathcal{C}}_n(ww') = \overline{\mathcal{C}}_n(w) \diamond \gamma^{-1}(w') + \gamma^{-1}(w) \diamond \overline{\mathcal{C}}_n(w')$$
for any $w, w' \in \mathfrak{H}$. The map $\overline{\mathcal{C}}_n$ is well-defined and $\overline{\mathcal{C}}_n(1) = 0$. Let $\overline{\rho}_n = M_n \overline{\mathcal{C}}_n$ $(n \geq 1)$. Then,

Lemma 3.1. *For any $n \geq 1$, we have $\rho_n = d\overline{\rho}_n$ on $\mathfrak{H}$.*

Proof. It suffices to show $\rho_n(w) = d\overline{\rho}_n(w)$ for $w = z_{k_1} \cdots z_{k_l} x^q$, where $q \geq 1$, $l \geq 0$ and $z_k = x^{k-1}y$ $(k \geq 1)$. By definition of $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$, we calculate

$$\mathcal{C}_n(w) = \sum_{p=1}^{q} x^{q-p+1} \otimes z^{\otimes(n-1)} \otimes z_{k_1} \cdots z_{k_l} z_p$$

$$+ \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} x^q \otimes z^{\otimes(n-1)} \otimes z_{k_1} \cdots z_{k_{j-1}} z_i$$

$$- \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} x^q \otimes z^{\otimes(n-1)} \otimes z_{k_1} \cdots z_{k_j}$$

and

$$\overline{\mathcal{C}}_n(w) = \sum_{p=1}^{q} \gamma^{-1}(x^{q-p+1}) \otimes y^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1} \cdots z_{k_l}) z_p$$

$$+ \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \gamma^{-1}(z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} x^q) \otimes y^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1} \cdots z_{k_{j-1}}) z_i$$

$$- \sum_{j=1}^{l} \gamma^{-1}(x z_{k_{j+1}} \cdots z_{k_l} x^q) \otimes y^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1} \cdots z_{k_{j-1}}) z_{k_j}.$$

According to the definition of the map d, we conclude $\rho_n(w) = d\overline{\rho}_n(w)$. $\square$

We define $\alpha \in \mathrm{Aut}(\mathfrak{H})$ by
$$\alpha(x) = y, \ \alpha(y) = x,$$
and a $\mathbb{Q}$-linear map $\widetilde{\alpha} \colon \mathfrak{H}^1 \to \mathfrak{H}^1$ by
$$\widetilde{\alpha}(wy) = \alpha(w)y \ (w \in \mathfrak{H}).$$

We easily find that

(8)
$$\varphi d = -d\widetilde{\alpha}.$$

Let $\overline{\ast} \colon \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ be a $\mathbb{Q}$-bilinear map defined by
  (i) $1 \overline{\ast} w = w \overline{\ast} 1 = w$ for any $w \in \mathfrak{H}^1$,
  (ii) $z_p w \overline{\ast} z_q w' = z_p(w \overline{\ast} z_q w') + z_q(z_p w \overline{\ast} w') - z_{p+q}(w \overline{\ast} w')$ for any $p, q \geq 1$ and any $w, w' \in \mathfrak{H}^1$,
which is known as an associative and commutative product on $\mathfrak{H}^1$, namely the harmonic product which is modeled by a series shuffle product rule of MZSV's. It is also known that the identity

(9)
$$d(w \overline{\ast} w') = d(w) \ast d(w')$$

holds for any $w, w' \in \mathfrak{H}^1$ (see [7, 8] for example).

The linear part of Kawashima relation for MZSV's proven in [7] is then stated as follows.

Theorem 3.2 (Kawashima). $L_x \widetilde{\alpha}(\mathfrak{H}y \overline{\ast} \mathfrak{H}y) \subset \ker \overline{Z}$.

We easily obtain the following equivalence.

Proposition 3.3. *For any $n \geq 1$, we have*
$$\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset L_x \widetilde{\alpha}(\mathfrak{H}y \overline{\ast} \mathfrak{H}y) \iff \rho_n(\check{\mathfrak{H}}^1) \subset L_x \varphi(\mathfrak{H}y \ast \mathfrak{H}y).$$

Proof. Assume that $\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset L_x \widetilde{\alpha}(\mathfrak{H}y \overline{\ast} \mathfrak{H}y)$. Using Lemma 3.1,
$$\rho_n(\check{\mathfrak{H}}^1) = d\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset d L_x \widetilde{\alpha}(\mathfrak{H}y \overline{\ast} \mathfrak{H}y).$$

Since the operators d and L_x commute,
$$\rho_n(\check{\mathfrak{H}}^1) \subset L_x d\widetilde{\alpha}(\mathfrak{H}y \overline{\ast} \mathfrak{H}y) = -L_x \varphi d(\mathfrak{H}y \overline{\ast} \mathfrak{H}y) = -L_x \varphi(d(\mathfrak{H}y) \ast d(\mathfrak{H}y)).$$

The first equality is by (8) and the second by (9). Therefore we have $\rho_n(\check{\mathfrak{H}}^1) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$ because of $d(\mathfrak{H}y) = \mathfrak{H}y$. In the same way, we can prove the reverse assertion. $\square$

According to this proposition, we find that CSF for MZV's is equivalent to that for MZSV's, which is also proven in Ihara-Kajikawa-Ohno-Okuda [5]. (Their proof is by direct calculation, which can also be applied for the q-analogue of MZV's.) Combining Theorem 3.2 with Proposition 3.3, we obtain

Corollary 3.4. *For any $n \geq 1$, we have $\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset \ker \overline{Z}$.*

Therefore, the operator $\overline{\rho}_n$ induces relations among MZSV's.

4. REMARKS

4.1. **Special evaluations.** We find that the following identities hold.

$$\rho_1(z_{k_1} \cdots z_{k_l}) = \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_j},$$

$$\overline{\rho}_1(\gamma(z_{k_1} \cdots z_{k_l}) - x^{k_1+\cdots+k_l}) = \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} z_i - k z_{k+1},$$

where $k = k_1 + \cdots + k_l$. Evaluating them by Z and $\overline{Z}$ respectively, we obtain the CSF's (1) and (2).

4.2. **An example.** Here we introduce another way to show CSF by means of the linear part of Kawashima relation in a special case. We first prove a lemma.

Lemma 4.1. *For any $q \geq 1, w \in \mathfrak{H}y, w' \in \mathfrak{H}^1$, we have*

$$zw * z_q w' = z(w * z_q w') + z_q(zw * w'),$$

where $z = x + y$ and $z_q = x^{q-1}y$ $(q \geq 1)$.

Proof. Let $w = z_p W$ $(p \geq 1)$. We see that

$$xw * z_q w' = z_{p+1} W * z_q w'$$
$$= z_{p+1}(W * z_q w') + z_q(z_{p+1} W * w') + z_{p+q+1}(W * w')$$

and

$$x(w * z_q w') = x(z_p W * z_q w')$$
$$= z_{p+1}(W * z_q w') + z_{q+1}(z_p W * w') + z_{p+q+1}(W * w').$$

By subtracting one from another, we obtain the identity

$$xw * z_q w' = x(w * z_q w') - z_{q+1}(w * w') + z_q(xw * w').$$

We also know that

$$yw * z_q w' = y(w * z_q w') + z_{q+1}(w * w') + z_q(yw * w').$$

Adding up these two identities, we have the lemma. $\square$

As a method to derive CSF from the linear part of Kawashima relation, we need to write down $\rho_n(z_{k_1} \cdots z_{k_l})$ explicitly in terms of the harmonic product. On the way, we find that the following identity holds.

Proposition 4.2. *For any $n, k \geq 1$, we have*

$$\rho_n(z_k) = L_x\varphi(A_{k-1} * z_n).$$

Proof. Note that the automorphism φ is an involution. Since

$$\mathcal{C}_n(z_k) = \sum_{i=1}^{k-1} z_{k-i+1} \otimes z^{\otimes(n-1)} \otimes z_i - x \otimes z^{\otimes(n-1)} \otimes z_k,$$

we have

$$\varphi L_x^{-1} \rho_n(z_k) = \sum_{i=1}^{k} A_{k-i} x^{n-1} A_i.$$

Hence, it is enough to show

$$(10) \qquad A_{k-1} * z_n = \sum_{i=1}^{k} A_{k-i} x^{n-1} A_i.$$

We prove (10) by induction on k. If $k = 1$, it holds because both sides become z_n. If $k = 2$,

$$\text{LHS} = y * z_n = y z_n + z_n y + z_{n+1},$$

$$\text{RHS} = A_1 x^{n-1} A_1 + x^{n-1} A_2 = y x^{n-1} y + x^{n-1} (x + y) y.$$

Therefore the identity (10) holds. Assume that (10) holds for $k - 1$ ($k \geq 2$). By Lemma 4.1,

$$A_{k-1} * z_n = z(A_{k-2} * z_n) + z_n A_{k-1}.$$

By the induction hypothesis,

$$A_{k-2} * z_n = \sum_{i=1}^{k-1} A_{k-i-1} x^{n-1} A_i,$$

and hence, we have

$$A_{k-1} * z_n = z \sum_{i=1}^{k-1} A_{k-i-1} x^{n-1} A_i + z_n A_{k-1}.$$

Notice that $z A_j = A_{j+1}$ ($j \geq 1$) but $z A_0 = A_1 + y$. Then, we have (10) and the proposition. □

When $k \geq 2$, the right-hand side of the identity of Proposition 4.2 is an element of $L_x \varphi(\mathfrak{H} y * \mathfrak{H} y)$. Therefore we have

$$\rho_n(z_k) \in \ker Z$$

according to Theorem 2.4. As an easy application, we also obtain the following.

Corollary 4.3. *For any $n, k \geq 1$, we have*

$$\rho_n(y z_k) = L_x \varphi(A_{k-1} * z_{n+1} - A_k * z_n).$$

Proof. The corollary is proven by Proposition 4.2 and the identity

$$(11) \qquad \rho_n(zw) = \rho_{n+1}(w) \ (w \in \mathfrak{H}).$$

□

This corollary and Theorem 2.4 yields $\rho_n(y z_k) \in \ker Z$ when $k \geq 2$.

4.3. **Dimentions.** We denote by $\check{\mathfrak{H}}^1_{(d)}$ the degree-d homogenous part of $\check{\mathfrak{H}}^1$. For $n, d \geq 1$, let

$$\mathrm{CSF}^n_d = \langle \rho_n(w) \mid w \in \check{\mathfrak{H}}^1_{(d)} \rangle_{\mathbb{Q}}, \ \mathrm{CSF}_d = \oplus_{n \geq 1} \mathrm{CSF}^n_d.$$

Then, we see the following filtration structure.

Proposition 4.4. *For any $n, d \geq 1$, we have $\mathrm{CSF}^{n+1}_d \subset \mathrm{CSF}^n_{d+1}$.*

Proof. The proof is just due to the identity (11). □

We obtain the following table of dimentions of CSF^n_d by calculation using Risa/Asir, an open source general computer algebra system.

weight $d + n$	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$n = 1$	1	2	4	6	12	18	34	58	106	186	350	$\cdots$
$n = 2$		1	3	5	11	17	33	57	105	185	349	$\cdots$
$n = 3$			1	3	7	13	26	48	91	167	319	$\cdots$
$n = 4$				1	3	7	15	29	58	111	218	$\cdots$
$n = 5$					1	3	7	15	31	61	122	$\cdots$
$n = 6$						1	3	7	15	31	63	$\cdots$
$n = 7$							1	3	7	15	31	$\cdots$
$n = 8$								1	3	7	15	$\cdots$
$n = 9$									1	3	7	$\cdots$
$n = 10$										1	3	$\cdots$
$n = 11$											1	$\cdots$

We also find that the sequence of $\dim_{\mathbb{Q}} \mathtt{CSF}_d$ corresponds to the sequence of $\dim_{\mathbb{Q}} \mathtt{CSF}_d^1$, which is the number of cyclic equivalent indices of weight k and depth $\leq k-1$ given by

$$-2 + \frac{1}{k} \sum_{m|k} \phi\left(\frac{k}{m}\right) 2^m,$$

where $k = d+1$ and $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^{\times}|$, Euler's totient function (see [10, Chap.1, Ex.27], [11, Chap.7, Ex.7.112] for example). We note that the same table can be obtained by considering $\mathbb{Q}$-vector spaces generated by CSF for MZSV's instead of CSF for MZV's.

4.4. Algebraic formulations of CSF and Derivation relation.

Let "$\cdot$" be an action of $\mathfrak{H}$ on $\mathfrak{H}^{\otimes(n+1)}$ defined by

$$a \cdot (w_1 \otimes \cdots \otimes w_{n+1}) = aw_1 \otimes w_2 \otimes \cdots \otimes w_{n+1},$$
$$(w_1 \otimes \cdots \otimes w_{n+1}) \cdot b = w_1 \otimes \cdots \otimes w_n \otimes w_{n+1}b.$$

The action "$\cdot$" is a $\mathfrak{H}$-bimodule structure on $\mathfrak{H}^{\otimes(n+1)}$ called the outer bimodule structure. For $n \geq 1$, we define the $\mathbb{Q}$-linear map $\mathcal{D}_n \colon \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\mathcal{D}_n(x) = x \otimes z^{\otimes(n-1)} \otimes y, \ \mathcal{D}_n(y) = -(x \otimes z^{\otimes(n-1)} \otimes y)$$

and

$$\mathcal{D}_n(ww') = \mathcal{D}_n(w) \cdot w' + w \cdot \mathcal{D}_n(w')$$

for any $w, w' \in \mathfrak{H}$. We find that the map $\mathcal{D}_n$ is well-defined. Let $\partial_n = M_n \mathcal{D}_n$. Then, we find that this ∂_n gives the derivation operator introduced in Ihara-Kaneko-Zagier [6], which induces Derivation relation for MZV's. Thus we find that there is a nice resemblance between algebraic formulations of CSF and Derivation relation.

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